[3]Shyy, W., Udaykumar, H. S., Rao, M. M., and Smith, R. W., *Computational Fluid Dynamics with Moving Boundaries*, Taylor and Francis, Philadelphia, 1996, pp. 31, 32.

[4]Jackson, P. S., and Christie, G. W., "Numerical Analysis of Three-Dimensional Elastic Membrane Wings," *AIAA Journal*, Vol. 25, 1987, pp. 676–682.

[5]Lian, Y., Shyy, W., Ifju, P. G., and Verron, E., "Membrane Wing Model for Micro Air Vehicles," *AIAA Journal*, Vol. 41, No. 12, 2003, pp. 2492–2494.

[6]Hutchison, M. G., Unger, E. R., Mason, W. H., Grossman, B., and Haftka, R. T., "Variable-Complexity Aerodynamic Optimization of a High-Speed Civil Transport Wing," *Journal of Aircraft*, Vol. 31, 1994, pp. 110–116.

[7]Burgee, S., Giunta, A. A., Narducci, R., Watson, L. T., Grossman, B., and Haftka, R. T., "A Coarse Grained Parallel Variable-Complexity Multidisciplinary Optimization Paradigm," *International Journal of Supercomputer Applications and High Performance Computing*, Vol. 10, 1996, pp. 269–299.

[8]Mason, B. H., Haftka, R. T., Johnson, E. R., and Farley, G. L., "Variable Complexity Design of Composite Fuselage Frames by Response Surface Techniques," *Thin Wall Structures*, Vol. 32, 1998, pp. 235–261.

[9]Knill, D. L., Giunta, A. A., Baker, C. A., Grossman, B., Mason, W. H., Haftka, R. T., and Watson, L.T., "Response Surface Methods Combining Linear and Euler Aerodynamics for Supersonic Transport Design," *Journal of Aircraft*, Vol. 36, 1999, pp. 75–86.

[10]*Design Optimization Tools*, Vanderplaats Research and Development, Inc., Colorado Springs, CO, 1999.

[11]Lian, Y., Shyy, W., and Haftka, R. T., "Shape Optimization of a Membrane Wing for Micro Air Vehicles," AIAA Paper 2003-0106, Jan. 2003.

[12]Duchon, J. P., "Splines Minimizing Rotation-Invariant Semi-Norms in Sobolev Spaces," *Constructive Theory of Functions of Several Variables*, edited by W. Schempp and K. Zeller, Springer-Verlag, Berlin, 1977, pp. 85–100.

A. Plotkin
Associate Editor

Benchmark Solution of Laminated Beams with Bonding Imperfections

W. Q. Chen,* J. Ying,† J. B. Cai,‡ and G. R. Ye‡
Zhejiang University, 310027 Hangzhou, People's Republic of China

Introduction

THE effect of interlaminar bonding imperfections on responses of laminated/sandwich beams, plates, or shells has received much attention recently. Most of the available works have employed various simplified beam, plate, or shell theories in which certain assumptions on the elastic fields along the thickness direction are introduced.[1–5] More recently, we presented a three-dimensional study of cross-ply laminated rectangular plates with imperfect interfaces, based on the so-called state-space formulations.[6] Through comparison, it was shown that the accuracy of the extended zig-zag plate theory[1] would become worse with the increasing imperfections of interfaces. In addition, the prediction of interfacial imperfections of a practical structure from the plate theory, when compared to the experimental results of deflection, will result in an underestimated value. This is usually not favorable to the practical damaged structures. Thus, although there are many highly accurate simplified theories or numerical methods for perfect laminated/sandwich beams,[7–10] it seems necessary to develop exact solutions that can be used as benchmarks for the analysis of imperfect laminated/sandwich beams.

This Note presents an exact solution of simply supported cross-ply laminated beams featuring interlaminar bonding imperfections described by a spring-layer model.[1–4] The analysis is similar to that presented in our previous work,[6] but the state-space formulations are established based on the two-dimensional elasticity equations for the plane-stress problem. The state-space approach is very effective in analyzing laminated beams because the scale of the final solving equations is independent of the number of layers. The numerical results presented in this Note should provide a useful means of comparison in the development of simplified theories for imperfect laminated/sandwich beam structures.

State-Space Approach for Plane-Stress Problem

An N-layered cross-ply laminated beam is shown in Fig. 1. The three-dimensional constitutive relations for a cross-ply laminate can be found in Ref. 6, for example. For a beam structure, because the width is very thin and the load along the width stays invariant, the problem can be regarded as a plane-stress problem.[11] In this case, the nonzero stress components are σ_x, σ_z, and τ_{xz} only, which are independent of y. Then we can derive the following two-dimensional constitutive relations:

$$\sigma_x = C_{11}\frac{\partial u}{\partial x} + C_{13}\frac{\partial w}{\partial z}, \qquad \sigma_z = C_{13}\frac{\partial u}{\partial x} + C_{33}\frac{\partial w}{\partial z}$$
$$\tau_{xz} = C_{55}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{1}$$

where u and w are the displacements in the x and z directions, respectively, and C_{ij} are the reduced stiffness constants, which can be expressed by the elastic constants c_{ij} as

$$C_{11} = c_{11} - c_{12}^2/c_{22}, \qquad C_{13} = c_{13} - c_{12}c_{23}/c_{22}$$
$$C_{33} = c_{33} - c_{23}^2/c_{22}, \qquad C_{55} = c_{55} \tag{2}$$

From Eq. (1) and the equations of motion,[11] the following state equation[12] can be obtained:

$$\frac{\partial}{\partial z}\begin{Bmatrix}\sigma_z\\ u\\ w\\ \tau_{xz}\end{Bmatrix} = \begin{bmatrix} 0 & 0 & \rho\dfrac{\partial^2}{\partial t^2} & -\dfrac{\partial}{\partial x}\\ 0 & 0 & -\dfrac{\partial}{\partial x} & \dfrac{1}{C_{55}}\\ \dfrac{1}{C_{33}} & -\dfrac{C_{13}}{C_{33}}\dfrac{\partial}{\partial x} & 0 & 0\\ -\dfrac{C_{13}}{C_{33}}\dfrac{\partial}{\partial x} & \rho\dfrac{\partial^2}{\partial t^2} - \alpha\dfrac{\partial^2}{\partial x^2} & 0 & 0 \end{bmatrix}\begin{Bmatrix}\sigma_z\\ u\\ w\\ \tau_{xz}\end{Bmatrix} \tag{3}$$

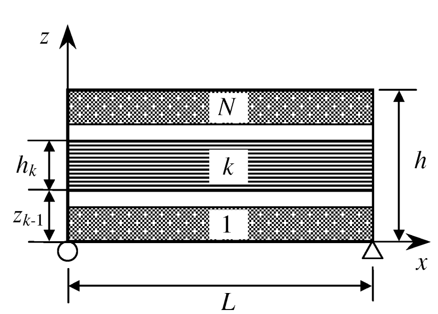


Fig. 1 Geometry and coordinates of a laminated beam.

Received 23 August 2003; revision received 13 September 2003; accepted for publication 15 September 2003.

*Professor, Department of Civil Engineering; chenwq@ccea.zju.edu.cn.
†Associate Professor, Department of Mechanical Engineering.
‡Associate Professor, Department of Civil Engineering.

where $\alpha = C_{11} - C_{13}^2/C_{33}$. In addition, we have

$$\sigma_x = \frac{C_{13}}{C_{33}}\sigma_z + \alpha\frac{\partial u}{\partial x} \tag{4}$$

For the simply supported boundary conditions, we can assume that

$$\begin{Bmatrix} \sigma_z \\ u \\ w \\ \tau_{xz} \end{Bmatrix} = \begin{Bmatrix} -C_{55}^{(1)}\bar{\sigma}_z(\zeta)\sin(m\pi\xi) \\ h\bar{u}(\zeta)\cos(m\pi\xi) \\ h\bar{w}(\zeta)\sin(m\pi\xi) \\ C_{55}^{(1)}\bar{\tau}_{xz}(\zeta)\cos(m\pi\xi) \end{Bmatrix} e^{i\omega t} \tag{5}$$

where $\xi = x/L$, $\zeta = z/h$, m is an integer, and ω is the circular frequency. The substitution of Eq. (5) into Eq. (3) yields

$$\frac{d}{d\zeta}\boldsymbol{V}(\zeta) = \boldsymbol{A}\boldsymbol{V}(\zeta) \tag{6}$$

where $\boldsymbol{V}(\zeta) = [\bar{\sigma}_z(\zeta), \bar{u}(\zeta), \bar{w}(\zeta), \bar{\tau}_{xz}(\zeta)]^T$ and the constant coefficient matrix $\boldsymbol{A}$ can be easily derived (omitted here for brevity). The solution to Eq. (6) can be easily obtained,[6] from which the following relation can be established:

$$\boldsymbol{V}_1^{(k)} = \boldsymbol{M}_k\boldsymbol{V}_0^{(k)} \tag{7}$$

where $\boldsymbol{V}_1^{(k)}$ and $\boldsymbol{V}_0^{(k)}$ are the state vectors at the upper and lower surfaces of the kth layer and $\boldsymbol{M}_k = e^{A(\zeta_k - \zeta_{k-1})}$ is the transfer matrix of that layer with $\zeta_k = z_k/h$.

If the imperfect interface is described by the general spring-layer model,[1,2] we obtain[6]

$$\boldsymbol{V}_0^{(k+1)} = \boldsymbol{P}_k\boldsymbol{V}_1^{(k)} \tag{8}$$

where $\boldsymbol{P}_k$ is the interfacial transfer matrix

$$\boldsymbol{P}_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \dfrac{C_{55}^{(1)}}{K_x^{(k)}h} \\ -\dfrac{C_{55}^{(1)}}{K_z^{(k)}h} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

with $K_i^{(k)}$ being the bonding stiffness constants of the interface between the kth layer and $(k+1)$th layer. The relation between state vectors at the top and bottom surfaces of the laminated beam is finally obtained from Eqs. (7) and (8) as

$$\boldsymbol{V}_1^{(N)} = \left(\prod_{k=N}^{2}\boldsymbol{M}_k\boldsymbol{P}_{k-1}\right)\boldsymbol{M}_1\boldsymbol{V}_0^{(1)} \tag{10}$$

Having established the preceding relation, we can easily deal with the bending and free vibration of a laminated beam, for which the analysis is very similar to that presented in Ref. 6 and that is omitted here for brevity.

Numerical Examples

For numerical calculation, it is assumed that each layer involved in the N-layered laminated beam has the same thickness and mass density. In addition, we assume that $K_z^{(k)} \to \infty$, that is, only slip-type interfacial imperfections are considered.[5] The following typical material properties are employed throughout this Note:

$$E_L/E_T = 25, \qquad G_{LT}/E_T = 0.5$$
$$G_{TT}/E_T = 0.2, \qquad \mu_{LT} = \mu_{TT} = 0.25 \tag{11}$$

where E is Young's modulus, G is shear modulus, μ is Poisson's ratio, and subscripts L and T indicate, respectively, directions parallel and perpendicular to the fibers.

First, consider the bending of a four-layered beam ([0/90/90/0 deg], stacking sequence from top to bottom), subjected to a normal sinusoidal pressure $q = q_0\sin(\pi\xi)$ at the top surface. The calculated results are given in Table 1, where a uniform imperfect bonding is assumed, that is, $R^{(k)} = E_T/[K_x^{(k)}h] = R$.

Note from Table 1 that the central deflection of the beam decreases with the increasing R, indicating that the interfacial imperfections will reduce the whole rigidity of the beam. The other variables shown in Table 1 also seem to vary with R monotonously; however, this will be different from point to point along the thickness direction. For illustration, the through-thickness distributions of normalized field variables for an asymmetric five-layered ([90/90/0/90/0 deg]) beam of $h/L = 0.2$ are given in Fig. 2. Nonuniform imperfections are assumed with $R^{(1)} = R^{(3)} = 0$ and $R^{(2)} = 3R^{(4)} = R$.

As shown in Fig. 2, unlike the beam deflection w, the displacement u and stresses τ_{xz} and σ_x all vary with R in a complicated way. Also note from Fig. 2c that the bonding imperfections reduce the transverse shear stress at the first and second interfaces. This indicates that the introduction of weak interfaces can be a useful means of preventing laminated beams from interlaminar shear failure. However, given that the rigidity of the beam decreases with R as shown in Fig. 2b, engineers should make a careful and exact evaluation of the effect of interfacial imperfections.

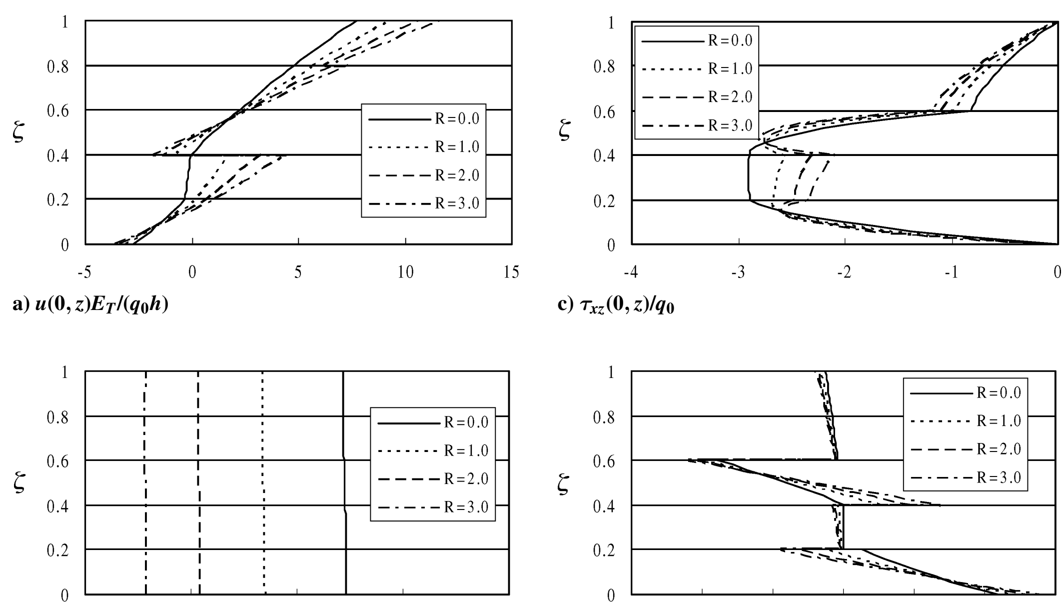
Table 1 Simply supported laminated beam ([0/90/90/0 deg]) with uniform interfacial imperfection under sinusoidal load

R	$[w(L/2, h/2)E_T]/q_0h$	$[u(0,0)E_T]/q_0h$	$[\sigma_x(L/2,0)]/q_0$	$[\tau_{xz}(0,h/2)]/q_0$
		$h/L = 0.25$		
0	−8.54838	−1.00096	19.6538	−1.45623
0.2	−9.78506	−1.09653	21.5303	−1.40861
0.4	−10.9424	−1.18597	23.2865	−1.36404
0.6	−12.0279	−1.26986	24.9336	−1.32223
		$h/L = 0.10$		
0	−103.749	−10.1243	79.5159	−4.02927
0.2	−113.293	−10.4495	82.0702	−4.00288
0.4	−122.713	−10.7705	84.5914	−3.97684
0.6	−132.011	−11.0874	87.0799	−3.95113
		$h/L = 0.05$		
0	−1089.29	−73.1069	287.090	−8.19843
0.2	−1128.89	−73.7916	289.779	−8.18451
0.4	−1168.35	−74.4740	292.459	−8.17063
0.6	−1207.68	−75.1540	295.129	−8.15680

Table 2 Lowest 10 frequency parameters $\omega^0 = \omega h\sqrt{(\rho/E_T)}$ of a simply supported laminated beam with nonuniform interfacial imperfections ($h/L = 0.1$, [0/90/0/0/90 deg])

Order	$R = 0$	$R = 0.3$	$R = 0.6$	$R = 0.9$	RE, %[a]
			$m = 1$		
1	0.0831525	0.0783998	0.0744920	0.0712077	14.36
2	1.20747	1.18895	1.16928	1.14857	4.88
3	2.16159	1.90554	1.74302	1.62897	24.64
4	3.16008	3.07005	2.79472	2.58585	18.17
5	3.62931	3.22634	3.18665	3.18136	12.34
6	5.50630	4.54124	3.89396	3.46330	37.10
7	6.39513	6.05482	5.26247	4.71476	26.28
8	7.29488	6.38877	6.38541	6.38405	12.49
9	9.37893	8.45264	7.95006	7.69748	17.93
10	9.54125	9.28972	8.58563	8.20946	13.96
			$m = 2$		
1	0.242910	0.219098	0.202498	0.190112	21.74
2	2.20588	2.02642	1.87061	1.74325	20.97
3	2.85496	2.61117	2.43824	2.29784	19.51
4	3.18242	3.17447	3.17102	3.16850	0.44
5	4.27129	3.96476	3.78270	3.66540	14.19
6	5.80075	4.95617	4.41315	4.08531	29.57
7	6.44563	6.36662	5.81370	5.36733	16.73
8	7.54524	6.50699	6.39770	6.38965	15.32
9	9.53048	8.53698	8.00206	7.73587	18.83
10	9.56825	9.38671	8.64412	8.24966	13.78

[a]Relative error $= (\omega^0|_{R=0} - \omega^0|_{R=0.9})/\omega^0|_{R=0}$.



a) $u(0,z)E_T/(q_0h)$

c) $\tau_{xz}(0,z)/q_0$

b) $w(L/2,z)E_T/(q_0h)$

d) $\sigma_x(L/2,z)/q_0$

Fig. 2 Distributions of normalized field variables along the thickness direction.

The first 10 lowest frequency parameters $\omega^0 = \omega h\sqrt{(\rho/E_T)}$ are given in Table 2 for a simply supported five-layered ([0/90/0/0/90 deg]) beam of $h/L = 0.1$. Nonuniform imperfections are assumed, that is, $R^{(1)} = 2R^{(2)} = R^{(3)} = R^{(4)} = R$. The reduction of rigidity of the beam with imperfect bonding is again verified by the results given in Table 2. The relative errors of frequency between the perfect beam and the imperfect beam with $R = 0.9$ are also calculated. Note that the reduction of frequency due to the bonding imperfections significantly depends on the frequency order. Thus, to improve the accuracy of health diagnosis of a laminated beam with interfacial imperfections, one should take into consideration a proper mode, of which the frequency is very sensitive to the bonding imperfections.

Conclusions

The two-dimensional state-space formulations are constructed for a simply supported cross-ply laminated beam featuring interlaminar bonding imperfections. No simplifying assumptions regarding the distributions of displacements or stresses along the thickness direction are introduced. Hence, the results presented in this Note should be useful for clarifying various beam theories or numerical methods.

If the buckling problem is considered, the following state equation can be derived from the basic equations

$$\frac{\partial}{\partial z}\begin{Bmatrix}\sigma_z\\ u\\ w\\ \tau_{xz}\end{Bmatrix} = \begin{bmatrix} 0 & 0 & \dfrac{p}{1+w^0_{,z}}\dfrac{\partial^2}{\partial x^2} & -\dfrac{\partial}{\partial x}\\ 0 & 0 & -\dfrac{\partial}{\partial x} & \dfrac{1}{C_{55}}\\ \dfrac{1}{C_{33}} & -\dfrac{C_{13}}{C_{33}}\dfrac{\partial}{\partial x} & 0 & 0\\ -\dfrac{C_{13}}{C_{33}}\dfrac{\partial}{\partial x} & \left(\dfrac{p}{1+u^0_{,x}}-\alpha\right)\dfrac{\partial^2}{\partial x^2} & 0 & 0 \end{bmatrix}\begin{Bmatrix}\sigma_z\\ u\\ w\\ \tau_{xz}\end{Bmatrix} \tag{12}$$

where p is the uniform compressive load applied at the two ends along the axial direction and

$$u^0_{,x} = -\frac{C_{33}p}{C_{11}C_{33}-C_{13}^2}, \qquad w^0_{,z} = \frac{C_{13}p}{C_{11}C_{33}-C_{13}^2} \tag{13}$$

are the normal strains in the prebuckling state. The buckling analysis for the nondebonded imperfect beams is then similar to that presented earlier for the free-vibration problem. In the case of a generally debonded beam, an exact buckling solution cannot be obtained and approximate treatments should be employed. However, if the beam is with through-length delamination at the kth interface, for example, an exact solution can be obtained, just as it can for the nondebonded imperfect beam, but with $K_x^{(k)} = 0$ and $K_z^{(k)} = 0$.

Finally, note that we have taken $K_z^{(k)} \to \infty$ in the numerical results to avoid the possibility of material penetration phenomenon.[1,2] In practice, if $K_z^{(k)}$ is finite, then the lower surface of the $(k+1)$th layer and the upper surface of the kth layer may be in contact, especially in a vibrating beam, making the problem nonlinear. In this case, it is generally impossible to derive the exact elasticity solution.

Acknowledgments

The work was supported by the National Natural Science Foundation of the People's Republic of China (Numbers 10002016 and 50105020).

References

[1] Cheng, Z. Q., Kennedy, D., and Williams, F. W., "Effect of Interfacial Imperfection on Buckling and Bending Behavior of Composite Laminates," *AIAA Journal*, Vol. 34, No. 12, 1996, pp. 2590–2595.

[2] Cheng, Z. Q., Jemah, A. K., and Williams, F. W., "Theory for Multilayered Anisotropic Plates with Weakened Interfaces," *Journal of Applied Mechanics*, Vol. 63, No. 4, 1996, pp. 1019–1026.

[3] Di Sciuva, M., "Geometrically Nonlinear Theory of Multilayered Plates with Interlayer Slips," *AIAA Journal*, Vol. 35, No. 11, 1997, pp. 1753–1759.

[4] Icardi, U., "Free Vibration of Composite Beams Featuring Interlaminar Bonding Imperfections and Exposed to Thermomechanical Loading," *Composite Structures*, Vol. 46, No. 3, 1999, pp. 229–243.

[5] Icardi, U., Di Sciuva, M., and Librescu, L., "Dynamic Response of Adaptive Cross-Ply Cantilevers Featuring Interlaminar Bonding Imperfections," *AIAA Journal*, Vol. 38, No. 3, 2000, pp. 499–506.

[6] Chen, W. Q., Cai, J. B., and Ye, G. R., "Exact Solutions of Cross-Ply Laminates with Bonding Imperfections," *AIAA Journal*, Vol. 41, No. 11, 2003, pp. 2244–2250.

[7] Perel, V. Y., and Palazotto, A. N., "Finite Element Formulations for Cylindrical Bending of a Transversely Compressible Sandwich Plate, Based on Transverse Strains," *International Journal of Solids and Structures*, Vol. 38, No. 30-31, pp. 5373–5409.

[8] Arya, H., Shimpi, R. P., and Naik, N. K., "Layer-by-Layer Analysis of a Simply Supported Thick Flexible Sandwich Beam," *AIAA Journal*, Vol. 40, No. 10, 2002, pp. 2133–2136.

[9] Huang, H. Y., and Kardomateas, G. A., "Buckling and Initial Postbuckling Behavior of Sandwich Beams Including Transverse Shear," *AIAA Journal*, Vol. 40, No. 11, 2002, pp. 2331–2335.

[10] Kapania, R. K., and Goyal, V. K., "Free Vibration of Unsymmetrically Laminated Beams Having Uncertain Ply Orientations," *AIAA Journal*, Vol. 40, No. 11, 2002, pp. 2336–2344.

[11] Timoshenko, S. P., and Goodier, J. N., *Theory of Elasticity*, 3rd ed., McGraw–Hill, New York, 1970, pp. 15–64.

[12] Bahar, L. Y., "A State Space Approach to Elasticity," *Journal of the Franklin Institute*, Vol. 299, No. 1, 1975, pp. 33–41.

A. Palazotto
Associate Editor

Shell Theory Accuracy with Regard to Initial Postbuckling Behavior of Cylindrical Shell

Izhak Sheinman* and Yiska Goldfeld†
Technion—Israel Institute of Technology,
32000 Haifa, Israel

Introduction

MANY studies concerned with the postbuckling behavior of cylindrical shells have been published in the past four decades. These studies were motivated by the fact that the behavior of cylindrical shells under buckling is characterized by the limit point rather than by the bifurcation point. Accordingly, the behavior is sensitive to initial imperfection; thus, a specific parameter in the relevant reduction factor, namely, the knockdown factor, acquires extreme importance. The factor is closely dependent on the postbuckling characteristic behavior.

When the behavior of structures characterized by the limit point is investigated, two main formulations exist:

The first is the quantitative approach, which consists in tracing the entire nonlinear equilibrium paths with emphasis on the level and direction of change of the stiffness during loading, as was done, for example, by Sheinman and Simitses[1] and Simitses et al.[2] (also the review paper of Simitses[3]). Under this approach, the complete nonlinear behavior is realized for a given imperfection shape and amplitude. It is extremely complicated and entails a heavy computational effort, and worst of all, it cannot cover all cases because each new configuration (of the geometry and/or of the imperfection) has to be reanalyzed from the beginning.

The second formulation is the qualitative approach, which consists in the parametric study of the shell in terms of its sensitivity to imperfection and its rating according to the postbuckling stiffness ratio, given by the initial change of stiffness slopes right at the bifurcation point. Koiter[4] was the first to show that the imperfection sensitivity of a structure is closely related to its initial postbuckling behavior, identified by the so-called Koiter ***b*** parameter. This was established in many well-known research works, for example, see Arbocz and Hol,[5] Budiansky,[6] and the review papers of Hutchinson and Koiter[7] and of Simitses.[3]

Most of the research concerning cylindrical shells used the simplest Donnell-type theory.[8] Comparison of different shell theories

Received 9 May 2003; revision received 21 July 2003; accepted for publication 1 October 2003.

*Professor, Faculty of Civil Engineering.

†Ph.D. Student, Faculty of Civil Engineering; currently Postdoctoral Fellow, Faculty of Aerospace Engineering, Delft University of Technology, P.O. Box 5058, 2600 GB Delft, The Netherlands.